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ANALYTICAL INVESTIGATION OF THE UNLOADING WAVE

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In the general case, the determination of the unloading wave shape [1] in the theory of elastic plastic wave propagation is reduced to the solution of a functional equation of complex structure. A characteristics method [2] is proposed for the approximate construction of the unloading wave, in particular loading cases formulas are obtained to determine its initial slope [3] and the next derivatives at the initial point [4 — 6]. An investigation of the general properties of an unloading wave is given in [7]. It is shown that as the load tends to zero asymptotically, the unloading wave at the end of a semi-infinite bar has an asymptote with a slope determined by the elastic wave velocity.

An investigation of the functional equation is given in this paper and a method of solution of this equation in the form of a power series is proposed. This approach to the problem permits obtaining both known and some new results. In the general loading case, formulas are obtained to determine the initial slope of the unloading wave and a method of determining the next derivatives at the initial point is indicated. Conditions are found for linear hardening for which the unloading wave is a straight line. The existence of an asymptote different from those mentioned in [7] is proved; it is shown how to continue the solution to adjacent sections by means of some known section, and an unloading wave in a material with delayed yielding is investigated.

1. System of functional equations for arbitrary hardening. Let us consider a semi-infinite bar $x \geqslant 0$ on whose end $x = 0$ a load is given in terms of the strain

$$u_x(0,\ t) = f(t) \tag{1.1}$$

$$f(0) \leqslant \varepsilon_s, \quad f'(t) \geqslant 0, \quad 0 \leqslant t \leqslant t_0; \quad f'(t) < 0, \quad t > t_0$$

Here $u(x,\ t)$ is the longitudinal displacement of the particle, $u_x = \partial u / \partial x$.

Let the dependence between the stress σ and the active strain ε be given as $\sigma = \Phi(\varepsilon)$, $\Phi'(\varepsilon) > 0$, $\Phi''(\varepsilon) < 0$, ε_s is the elastic strain limit. A linear dependence $\sigma - \sigma_0 = E(\varepsilon - \varepsilon_0)$, where E is Young's modulus and $\sigma_0 = \Phi(\varepsilon_0)$, is taken for unloading from the strain $\varepsilon_0 > \varepsilon_s$.

In the region of active elastic-plastic strains the solution of the equation of motion

$$a^2 u_{xx} = u_{tt} \qquad (a^2 = \Phi'(\varepsilon) / \rho)$$

with zero initial data and the boundary condition (1.1) is represented in implicit form as

$$u_x = f\left(t - \frac{x}{a(u_x)}\right), \qquad u_t = -\int_0^{u_x} a(\varepsilon)\, d\varepsilon \equiv -\psi(u_x) \tag{1.2}$$

If the unloading wave $x = \varphi(t)$ (for $\varphi(t_0) = 0$) is the front of a strong discontinuity, then it is a straight line $x = a_0(t - t_0)$; $a_0^2 = E / \rho$ [8, 9]. Here, the most interesting case of a curvilinear unloading wave, which is the front of a weak discontinuity, is investigated.

In the passive strain region $x < \varphi(t)$, $t > t_0$ the motion parameters are determined from the solution of the problem

$$u_{tt} = a_0^2 u_{xx} + \frac{1}{\rho}\frac{d\sigma_0}{dx} - a_0^2 \frac{d\varepsilon_0}{dx} \tag{1.3}$$

$$u_x(0,\ t) = f(t) \tag{1.4}$$

$$u_x^{(2)} = u_x^{(1)} \equiv \varepsilon_0(x), \quad u_t^{(2)} = u_t^{(1)}, \quad x = \varphi(t) \tag{1.5}$$

where the superscripts 1 and 2 refer to the active and passive strain domains, respectively, $\sigma_0(x) = \Phi(\varepsilon_0)$ and $\varepsilon_0(x)$ are values of the stress and strain on the unloading wave.

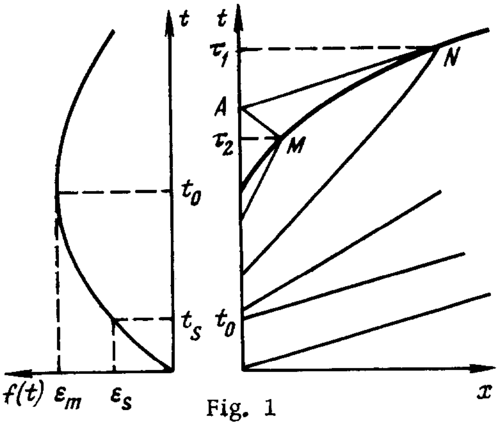

Fig. 1

Let us consider the curvilinear triangle AMN (Fig. 1), where $N(\varphi(\tau_1),\ \tau_1)$ and $M(\varphi(\tau_2),\ \tau_2)$ are the points of intersection of the unloading wave shown by the heavy line, and the characteristics of positive and negative slope emerging from some point A $(0,\ t)$ for $t > t_0$. Along the characteristics of Eq. (1.3) with regard to (1.5), the following relations can be obtained:

$$\sigma_0(\varphi(\tau_1)) / \rho a_0 + \psi(\varepsilon_0[\varphi(\tau_1)]) = a_0 u_x - u_t + (\sigma_0 - E\varepsilon_0) / \rho a_0 \tag{1.6}$$

$$\sigma_0(\varphi(\tau_2)) / \rho a_0 - \psi(\varepsilon_0[\varphi(\tau_2)]) = a_0 u_x + u_t + (\sigma_0 - E\varepsilon_0) / \rho a_0 \tag{1.7}$$

Applying the equalities (1.6) and (1.7) at the common point A, where $u_x = f(t)$, and adding them ,with (1.2) taken into account for u_t, we obtain

$$\int_0^{\varepsilon_0(\varphi(\tau_1))} \left[\frac{a^2(\xi)}{a_0} + a(\xi)\right] d\xi + \int_0^{\varepsilon_0(\varphi(\tau_2))} \left[\frac{a^2(\xi)}{a_0} - a(\xi)\right] d\xi = \qquad (1.8)$$

$$2a_0 f(t) + \frac{2}{\rho a_0}[\Phi(\varepsilon_m) - E\varepsilon_m]$$

Here $\varepsilon_m = \varepsilon_0(0) = f(t_0)$ is the maximum strain at the end of the bar.

According to (1.2), the conditions (1.5) in application to the points N and M, become

$$\varepsilon_0(\varphi(\tau_1)) = f\left(\tau_1 - \frac{\varphi(\tau_1)}{a(\varepsilon_0[\varphi(\tau_1)])}\right) \qquad (1.9)$$

$$\varepsilon_0(\varphi(\tau_2)) = f\left(\tau_2 - \frac{\varphi(\tau_2)}{a(\varepsilon_0[\varphi(\tau_2)])}\right) \qquad (1.10)$$

Finally, according to definition (Fig. 1), we have

$$\tau_1 - t = \varphi(\tau_1) / a_0, \quad t - \tau_2 = \varphi(\tau_2) / a_0 \qquad (1.11)$$

We shall consider the equalities (1.8) — (1.11) as a system of functional equations in the unknown functions $\varphi(t)$, $\tau_1(t)$, $\tau_2(t)$ and $\varepsilon_0(x)$. Hence, (1.9) and (1.10) are substantially identical and written differently for convenience. The solution of the system (1.8) — (1.11) determines the unloading wave $x = \varphi(t)$ and the value of the strain $\varepsilon_0(x)$ thereon.

We investigate the asymptotic behvior of the unloading wave. Let us note that the right side in (1.8) equals $2p(t) / \rho a_0$, where $p(t)$ is the stress at the end of the bar.

As $\tau_1 \to \infty$ let the quantities t and τ_2 tend to infinity. Then $\varepsilon_0(\varphi(\tau_1))$ and $\varepsilon_0(\varphi(\tau_2))$ tend to a finite limit $\varepsilon_* \geqslant \varepsilon_s$. Passing to the limit in (1.8), we obtain

$$\rho \int_0^{\varepsilon_*} a^2(\xi)\, d\xi = p(\infty)$$

If $0 \leqslant p(\infty) < \sigma_s$, then the equality obtained is contradictory, hence as $\tau_1 \to \infty$ the quantity t tends to the finite limit t_* and the straight line $x = a_0(t - t_*)$ is the asymptote to which the curve $x = \varphi(t)$ tends from below. Hence $\varepsilon_* = \varepsilon_s$ on the unloading wave at infinity. This case is similar to that examined in [7].

If $p(\infty) \geqslant \sigma_s$, then it follows from the equality obtained that $\sigma_* = p(\infty)$, where σ_* is the stress on the unloading wave at infinity. In this case the unloading wave tends to the asymptote $x = a(f(t_{**}))(t - t_{**})$ from above, where $t_{**} < t_0$ is the instant at which $p(t_{**}) = p(\infty)$.

2. Determination of the initial unloading wave velocity. The initial velocity of unloading wave propagation $c_1 = \varphi'(t_0)$ depends on the behavior of the load function $f(t)$ in the neighborhood of the beginning of the unloading. Let us examine several cases.

1) Let t_0 be a point of discontinuity of $f'(t)$, i.e.

$$f'(t_0 - 0) = \alpha_1 > 0, \quad f'(t_0 + 0) = \alpha_2 < 0$$

Differentiating (1.8) with respect to t we obtain

$$\left[\frac{b^2(\tau_1)}{a_0}+b(\tau_1)\right]\varepsilon_0'(\varphi(\tau_1))\varphi'(\tau_1)\frac{d\tau_1}{dt}+\left[\frac{b^2(\tau_2)}{a_0}-b(\tau_2)\right]\times \tag{2.1}$$

$$\varepsilon_0'(\varphi(\tau_2))\varphi'(\tau_2)\frac{d\tau_2}{dt}=2a_0f'(t),\quad b(\tau)=a(\varepsilon_0[\varphi(\tau)])$$

The derivatives ε_0', $d\tau_1/dt$, $d\tau_2/dt$ in (2.1) are determined from the relationships (1.9)—(1.11) differentiated with respect to t. The relationships

$$\varphi(\tau_1)\to 0,\quad \varphi(\tau_2)\to 0,\quad \varepsilon_0(\varphi(\tau_1))\to f(t_0),\quad \varepsilon_0(\varphi(\tau_2))\to f(t_0)$$

$$b(\tau_1)\to a(f(t_0))\equiv a_p,\quad b(\tau_2)\to a(f(t_0))\equiv a_p$$

$$\tau_1-\frac{\varphi(\tau_1)}{b(\tau_1)}\to t_0-0,\quad \tau_2-\frac{\varphi(\tau_2)}{b(\tau_2)}\to t_0-0$$

$$\frac{d\varepsilon_0(\varphi(\tau_2))}{dt}\to\alpha_1\frac{a_0}{a_p}\frac{a_p-c_1}{a_0+c_1}$$

$$\frac{d\varepsilon_0(\varphi(\tau_1))}{dt}\to\alpha_1\frac{a_0}{a_p}\frac{a_p-c_1}{a_0-c_1},\quad c_1\neq a_0$$

hold in these differentiated equalities for t, τ_1, τ_2, tending to t_0+0.

Letting t tend to t_0+0 in (2.1) and taking account of the limit values obtained, we find

$$(a_p+a_0)\frac{a_p-c_1}{a_0-c_1}+(a_p-a_0)\frac{a_p-c_1}{a_0+c_1}=\frac{2a_0}{m}\quad\left(m=\frac{\alpha_1}{\alpha_2}\right) \tag{2.2}$$

$$c_1=[(a_0^2-ma_p^2)/(1-m)]^{1/2}$$

This formula has been obtained by another method in [3, 8, 9].

2) We construct an equation for c_1 in the case when

$$f^{(n)}(t_0+0)=f^{(n)}(t_0-0)\neq 0,\quad f^{(p)}(t_0)=0,\quad 1\leqslant p\leqslant n-1$$

where n is even and $f^{(n)}(t_0)<0$, since $f(t)$ has a maximum at $t=t_0$.

Differentiating the equalities (1.8)—(1.11) n times with respect to t and passing to the limit as $t\to t_0+0$, we obtain similarly to the preceding, an equation of degree $2n$ in c_1

$$P(c_1)\equiv\left(\frac{a_0}{a_p}\right)^{n-1}(a_p-c_1)^n[(a_0+a_p)(a_0+c_1)^n- \tag{2.3}$$

$$(a_0-a_p)(a_0-c_1)^n]-2a_0(a_0^2-c_1^2)^n=0$$

The quantity c_1 satisfies the inequalities $a_p\leqslant c_1\leqslant a_0$, hence $P(a_p)<0$, $P(a_0)>0$ and $P'(c_1)>0$. Consequently, Eq. (2.3) has one real root in the interval (a_p, a_0).

3) Now let us consider the case when $f^{(n)}(t)$ has a discontinuity at the point t_0

$$f^{(n)}(t_0-0)=\alpha\neq 0,\quad f^{(n)}(t_0+0)=\beta\neq 0$$

$$f^{(p)}(t_0)=0,\quad 1\leqslant p\leqslant n-1$$

Here $\alpha<0$, $\beta<0$ for n even and $\alpha>0$, $\beta<0$ for n odd. Analogously to the preceding, we obtain an equation for c_1

$$\alpha\left(\frac{a_0}{a_p}\right)^n\left[\left(\frac{a_p-c_1}{a_0-c_1}\right)^n-\frac{a_0-a_p}{a_0+a_p}\left(\frac{a_p-c_1}{a_0+c_1}\right)^n\right]=\frac{2a_0^2\beta}{a_p(a_0+a_p)} \tag{2.4}$$

which has one real root in the interval (a_p, a_0) and yields the value (2.2) for c_1 at $n=1$.

If the stress $p(t)$ is given at the end of the bar and this function is of the form described above in the analysis of the 2nd case, then because of the difference in the $\sigma \sim \varepsilon$ coupling laws for loading and unloading, the function $f(t)$ has a discontinuous n-th derivative, where $\alpha a_p^2 = \beta a_0^2$. Hence we obtain the equation

$$\left(\frac{a_0}{a_p}\right)^n \left[\left(\frac{a_p - c_1}{a_0 - c_1}\right)^n - \frac{a_0 - a_p}{a_0 + a_p}\left(\frac{a_p - c_1}{a_0 + c_1}\right)^n\right] = \frac{2a_p}{a_0 + a_p} \qquad (2.5)$$

from (2.4) to determine c_1, which becomes for $n = 2$

$$c_1^2 (a_p c_1^2 + 2a_0^2 c_1 - 3a_0^2 a_p) = 0$$

A positive root of this equation is

$$c_1 = a_0\left[\left(\frac{a_0^2}{a_p^2} + 3\right)^{1/2} - \frac{a_0}{a_p}\right]$$

This formula has been obtained by another method in [3, 9].

4) Let $f^{(n)}(t_0 - 0) = \alpha \neq 0$, $f^{(p)}(t_0 - 0) = 0$ for $1 \leqslant p \leqslant n - 1$, where $\alpha < 0$ for n even, $\alpha > 0$ for n odd, and $f^{(k)}(t_0 + 0) \equiv \beta < 0$, $f^{(p)}(t_0 + 0) = 0$ for $1 \leqslant p \leqslant k - 1$.

a) If $k = n$, then c_1 is determined from (2.4).

b) If $k > n$, then we have the equation

$$\left(\frac{a_p - c_1}{a_0 - c_1}\right)^n - \frac{a_0 - a_p}{a_0 + a_p}\left(\frac{a_p - c_1}{a_0 + c_1}\right)^n = 0$$

to determine c_1, from which $c_1 = a_p$.

c) Let $k < n$. Let us note that all the equations obtained above to determine $c_1 = \varphi'(t_0)$ have been derived under the assumption that $c_1 \neq a_0$. This assumption leads to a contradiction in the case under consideration. In fact, by differentiating the equalities (1.8) — (1.11) k times with respect to t and passing to the limit as $t \to t_0 + 0$, we obtain a contradictory equality. Therefore, $c_1 = a_0$ in this case.

3. Analytical determination of the unloading wave. We assume that the function $\varphi(t)$ is representable as the Taylor series

$$\varphi(t) = \sum_{n=1}^{\infty} \frac{\varphi^{(n)}(t_0)}{n!}(t - t_0)^n$$

Let us show that by differentiating the functional equations (1.8) — (1.11) successively with respect to t and passing to the limit as $t \to t_0 + 0$, the values of the derivatives $\varphi^{(n)}(t_0)$ of any order can be calculated.

We introduce the notation

$$\varepsilon_{01} \equiv \varepsilon_0(\varphi(\tau_1)), \qquad \varepsilon_{02} \equiv \varepsilon_0(\varphi(\tau_2))$$

$$A_1(\varepsilon_{01}) = \int_0^{\varepsilon_{01}} \left(\frac{a^2(\xi)}{a_0} + a(\xi)\right) d\xi, \quad A_2(\varepsilon_{02}) = \int_0^{\varepsilon_{02}} \left(\frac{a^2(\xi)}{a_0} - a(\xi)\right) d\xi$$

$$A_1^{(p)} = d^p A_1 / d^p \varepsilon_{01}, \quad A_2^{(p)} = d^p A_2 / d\varepsilon_{02}^p$$

Differentiating (1.8) p times with respect to t, we find

$$\sum_{k=1}^{p}\left(A_1^{(k)}\frac{U_{1k}}{k!}+A_2^{(k)}\frac{U_{2k}}{k!}\right)=2a_0f^{(p)}(t) \qquad (3.1)$$

$$U_{j1}=\frac{d^p}{dt^p}\varepsilon_{0j},\ \ldots,\qquad U_{jk}=\sum_{i=0}^{k-1}(-1)^i\binom{k}{i}\varepsilon_{0j}{}^i\frac{d^p}{dt^p}\varepsilon_{0j}^{k-i}\ ,\ \ldots$$

$$\ldots,U_{jp}=\left(\frac{d\varepsilon_{0j}}{dt}\right)^p;\qquad \binom{k}{i}=\frac{k(k-1)\ldots(k-i+1)}{i!},\quad j=1,2$$

It is here understood that the hardening curve $\Phi(\varepsilon)$ is sufficiently smooth and generally has derivatives of any order for all values $\varepsilon>\varepsilon_s$.
We use the notation

$$Q_1=\tau_1-\frac{\varphi(\tau_1)}{a_1},\quad Q_2=\tau_2-\frac{\varphi(\tau_2)}{a_2},\quad Q_i^{(p)}=\frac{d^pQ_i}{dt^p}$$

$$a_1=a(\varepsilon_{01}),\ a_2=a(\varepsilon_{02}),\quad a_i^{(p)}=\frac{d^pa_i}{d\varepsilon_{0i}^p}$$

Utilizing the expressions for $d\tau\ /\ dt$ and $d\tau_2\ /\ dt$ from (1.11) to evaluate the derivatives of Q_1 and Q_2 with respect to t, we find

$$Q_1'=\frac{a_0}{a_1}\frac{a_1-\varphi'(\tau_1)}{a_0-\varphi'(\tau_1)}+\varphi(\tau_1)\frac{a_1'}{a_1^2}\frac{d\varepsilon_{01}}{dt} \qquad (3.2)$$

$$Q_1''=\frac{a_0^2}{a_1}\frac{(a_1-a_0)\varphi''(\tau_1)}{[a_0-\varphi'(\tau_1)]^3}+2\frac{a_0a_1'}{a_1^2}\frac{\varphi'(\tau_1)}{[a_0-\varphi'(\tau_1)]}\frac{d\varepsilon_{01}}{dt}+$$
$$\varphi(\tau_1)\frac{a_1a_1''-2a_1'^2}{a_1^3}\left(\frac{d\varepsilon_{01}}{dt}\right)^2+\varphi(\tau_1)\frac{a_1'}{a_1^2}\frac{d^2\varepsilon_{01}}{dt^2}$$

$$Q_1'''=\frac{a_0^3}{a_1}\frac{(a_1-a_0)\varphi'''(\tau_1)}{[a_0-\varphi'(\tau_1)]^4}+3\frac{a_0^3}{a_1}\frac{(a_1-a_0)[\varphi''(\tau_1)]^2}{[a_0-\varphi'(\tau_1)]^5}+\ldots+$$
$$\varphi(\tau_1)\frac{a_1'}{a_1^2}\frac{d^3\varepsilon_{01}}{dt^3}$$

. . . .

$$Q_2'=\frac{a_0}{a_1}\frac{a_2-\varphi'(\tau_2)}{a_0+\varphi'(\tau_2)}+\varphi(\tau_2)\frac{a_2'}{a_2^2}\frac{d\varepsilon_{02}}{dt}$$

. . . .

Here terms containing the factors $d\varepsilon_{01}\ /\ dt$ and $d^2\varepsilon_{01}\ /\ dt^2$ have been omitted in the expressions for Q_1'''.

Differentiating the equalities (1.9) and (1.10) with respect to t, we obtain successively

$$d\varepsilon_{01}\ /\ dt=f'(Q_1)Q_1' \qquad (3.3)$$

$$\frac{d^2\varepsilon_{01}}{dt^2}=f''(Q_1)Q_1'^2+f'(Q_1)Q_1''$$

$$\frac{d^3\varepsilon_{01}}{dt^3}=f'''(Q_1)Q_1'^3+3f''(Q_1)Q_1'Q_1''+f'(Q_1)Q_1'''$$

. . . .

$$\frac{d^p\varepsilon_{01}}{dt^p}=f^{(p)}(Q_1)Q_1'^p+g_{1p}f^{(p-1)}(Q_1)Q_1'^{p-2}Q_1''+\ldots+f'(Q_1)Q_1^{(p)}$$

$$d\varepsilon_{02}/dt=f'(Q_2)Q_2'$$

. . . .

Let us examine the 2nd case described in Sect. 2 in detail. The reasoning for the remaining cases (except the case $c_1 = a_0$) is similar. Passing to the limit for t, τ_1 and τ_2 tending to $t_0 + 0$ in (3.2) and (3.3), we see that all the derivatives of ε_{01} and ε_{02} with respect to t at the point t_0, to the $(n-1)$-th order inclusive are zero, and we obtain for the n-th order derivatives

$$\left.\frac{d^n\varepsilon_{01}}{dt^n}\right|_{t_0} = f^{(n)}(t_0)\left[\frac{a_0}{a_p}\,\frac{a_p - c_1}{a_0 - c_1}\right]^n \qquad (3.4)$$

$$\left.\frac{d^n\varepsilon_{02}}{dt^n}\right|_{t_0} = f^{(n)}(t_0)\left[\frac{a_0}{a_p}\,\frac{a_p - c_1}{a_0 + c_1}\right]^n$$

Therefore, equalities of the form (3.1) to $p = n - 1$ inclusive, become identities under the mentioned passage to the limit, and for $p = n$, after having divided by $f^{(n)}(t_0)$ we arrive at (2.3) with respect to the first derivative $c_1 = \varphi'(t_0)$.

Now, applying (3.3) for $p = n + 1$, we see that the limit values of the derivatives are

$$\left.\frac{d^{n+1}\varepsilon_{01}}{dt^{n+1}}\right|_{t_0} = f^{(n+1)}(t_0)\,Q'^{\,n+1}_{10} + g_{1n}f^{(n)}(t_0)\,Q'^{\,n-1}_{10}Q_{10}'' \qquad (3.5)$$

$$\left.\frac{d^{n+1}\varepsilon_{02}}{dt^{n+1}}\right|_{t_0} = f^{(n+1)}(t_0)\,Q'^{\,n+1}_{20} + g_{2n}f^{(n)}(t_0)\,Q'^{\,n-1}_{20}Q_{20}'' \qquad (3.6)$$

where Q_{10}' and Q_{20}' contain the value already found for c_1 according to (3.2), and Q_{10}'' and Q_{20}'' contain in addition to c_1 only $c_2 = \varphi''(t_0)$ (but linearly). Therefore, an equality of the type (3.1) for $p = n + 1$ yields a linear algebraic equation in c_2 in the limit as $t \to t_0 + 0$. Similarly, for $p = n + 2$ we obtain a linear inhomogeneous algebraic equation on $c_3 = \varphi'''(t_0)$.

Therefore, successive use of limit equalities of the type (3.1) — (3.3) permits the evaluation of any derivative $\varphi^{(q)}(t_0)$, i.e. the determination of any number of terms in the Taylor series. Analogous results have been obtained in [4] for the case when the curve $f(t)$ has a salient point t_0.

For linear hardening, when $\sigma = E_1\varepsilon + (E - E_1)\varepsilon_s$ for $\varepsilon > \varepsilon_s$, the system of functional equations (1.8) — (1.11) reduces to the following:

$$(a_0 + a_1) f\left(\tau_1 - \frac{\varphi(\tau_1)}{a_1}\right) - (a_0 - a_1) f\left(\tau_2 - \frac{\varphi(\tau_2)}{a_1}\right) = \qquad (3.7)$$

$$\frac{2a_0^2}{a_1} f(t) + 2\frac{a_1^2 - a_0^2}{a_1} f(t_0)$$

$$\tau_1 - t = \varphi(\tau_1)/a_0, \quad t - \tau_2 = \varphi(\tau_2)/a_0 \qquad (3.8)$$

where $a_1 = (E_1/\rho)^{1/2}$ is the velocity of plastic strain propagation.

It is clear that the mentioned procedure to determine the derivatives hence remains substantially unchanged and is simplified because a_1 is a constant. Formulas (2.2) — (2.6) retain their form with the replacement of a_p by a_1. The linear hardening case has been investigated by another method in [6].

4. Particular loading cases. We examine three particular cases of giving the loading curve at the end of the bar $f(t)$.

1) Let $f(t)$ be a parabola of n-th degree

$$f(t) = f(t_0) + \frac{f^{(n)}(t_0)}{n!}(t - t_0)^n, \qquad f^{(n)}(t_0) \neq 0,\ f^{(p)}(t_0) = 0,\ p \neq n$$

Hence, equations of the form (3. 1) with linear hardening become identities for $p \leqslant n - 1$ in the limit as $t \to t_0 + 0$. According to (3. 2) and (3. 3), for $p = n$ we obtain Eq. (2. 3) for $c_1 = \varphi'(t_0)$. Finding c_1 from it and writing (3. 1) for $p = n + 1$ in the limit as $t \to t_0$, we see that according to (3. 6), a linear homogeneous equation is obtained for $c_2 = \varphi''(t_0)$, i. e. $c_2 = 0$. Similarly, we find that $\varphi^{(k)}(t_0) = 0$ for $k > 1$.

Therefore, the unloading wave is a straight line in this case, whose slope is determined from (2. 3).

2) Let

$$f(t) = \begin{cases} f(t_0) + \dfrac{\alpha}{n!}(t - t_0)^n, & t \leqslant t_0 \\ f(t_0) + \dfrac{\beta}{n!}(t - t_0)^n, & t \geqslant t_0 \end{cases}$$

where $\alpha < 0, \beta > 0$ for n even and $\alpha > 0, \beta < 0$ for n odd. Analogously to the above, we find that the unloading wave is a straight line. The solution in the form of a line has been found in [6] for $n = 1$.

Investigations of cases 1 and 2 refer only to that initial part of the unloading wave which corresponds to a decrease in the load to zero at the bar end. The case of loads of opposite sign is not considered. If the load vanishes for $t = t_k > t_0$, and then remains zero, then the unloading wave remains a straight line to the instant of time t_p determined by the intersection of the lines $x = c_1(t - t_0)$ and $x = a_0(t - t_k)$, i. e. to $t_p = (a_0 t_k - c_1 t_0) / (a_0 - c_1)$.

3) For an idealized explosive type load ($f(0) = \varepsilon_m > \varepsilon_s$, $f'(t) < 0$) the unloading wave passes through the origin [9] of the (x, t) plane and the Riemann wave domain in which we have for the stretching effect

$$u_x = a^{-1}(x / t), \quad u_t = -\psi(u_x)$$

is adjoining.

The functional equations become

$$\int_0^{a^{-1}(\varphi(\tau_1)/\tau_1)} \left[\frac{a^2(\xi)}{a_0} + a(\xi)\right] d\xi + \int_0^{a^{-1}(\varphi(\tau_2)/\tau_2)} \left[\frac{a^2(\xi)}{a_0} - a(\xi)\right] d\xi = 2a_0 f(t) + \frac{2}{\rho a_0}[\Phi(\varepsilon_m) - E\varepsilon_m] \tag{4. 1}$$

$$\tau_1 - t = \varphi(\tau_1) / a_0, \quad t - \tau_2 = \varphi(\tau_2) / a_0 \tag{4. 2}$$

Passing to the limit as $t \to 0$ in (4. 1) and taking into account that $\varphi(\tau_1) / \tau_1 \to \varphi'(0)$, $\varphi(\tau_2) / \tau_2 \to \varphi'(0)$, we obtain $c_1 \equiv \varphi'(0) = a(\varepsilon_m)$. We differentiate (4. 1) with respect to t and we pass to the limit as $t \to 0$. Taking into account that

$$\frac{d}{d\tau_1}\left(\frac{\varphi(\tau_1)}{\tau_1}\right) \to \frac{\varphi''(0)}{2}, \quad \frac{d}{d\tau_2}\left(\frac{\varphi(\tau_2)}{\tau_2}\right) \to \frac{\varphi''(0)}{2}$$

$$a^{-1'}\left(\frac{\varphi(\tau_1)}{\tau_1}\right) \to \frac{1}{a'(\varepsilon_m)}, \quad a^{-1'}\left(\frac{\varphi(\tau_2)}{\tau_2}\right) \to \frac{1}{a'(\varepsilon_m)}$$

(the prime denotes the derivative with respect to the argument in the parentheses), we obtain

$$\left(\frac{c_1^2}{a_0} + c_1\right)\frac{c_2 a_0}{2(a_0 - c_1)} + \left(\frac{c_1^2}{a_0} - c_1\right)\frac{c_2 a_0}{2(a_0 + c_1)} = 2a_0 f'(0)\, a'(\varepsilon_m)$$

$$c_2 = f'(0)\, a'(\varepsilon_m) \frac{a_0^2 - a^2(\varepsilon_m)}{a^2(\varepsilon_m)}, \qquad c_3 \equiv \varphi''(0)$$

Higher order derivatives can be evaluated in a similar manner. Analogous results have been obtained in [5] for this case.

5. Continuation of the solution. Limiting ourselves to linear hardening to keep the writing short, i.e. to the system of functional equations (3.7) and (3.8), we note that the results obtained below can be carried over to the case of arbitrary hardening without difficulty, i.e. to the systems (1.8) — (1.11) or (4.1), (4.2). Each of these systems connects the values of τ_2 and $\varphi(\tau_2)$ to the values of τ_1 and $\varphi(\tau_1)$. Hence, if the value of $\varphi(\tau_2)$ has been determined at the point B (Fig. 2), then the value of $\varphi(\tau_1)$ can be obtained at the point D by a closure through the point C. Therefore, if the solution $x = \varphi(t)$ is known on the section BD, cut off by characteristics with positive and negative slope drawn from some point C of the time axis, then it can be continued at both sides. For instance, we continue the solution of the system (3.7), (3.8) upward expressing $\varphi(\tau_1)$ and τ_1 from (3.7) and (3.8) in terms of $\varphi(\tau_2)$ and τ_2

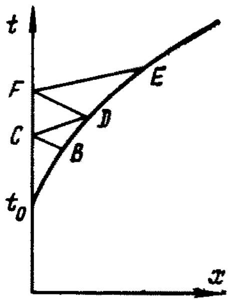

Fig. 2

$$\tau_1 = \frac{a_0\tau_2 + \varphi(\tau_2)}{a_0 - a_1} + \xi \tag{5.1}$$

$$\varphi(\tau_1) = a_0 \frac{a_0\tau_2 + \varphi(\tau_2)}{a_0 - a_1} - \varphi(\tau_2) - a_0\tau_2 + a_0\xi$$

$$\xi = \frac{a_1}{a_1 - a_0} f^{-1}\left\{\frac{a_0 - a_1}{a_0 + a_1} f\left(\tau_2 - \frac{\varphi(\tau_2)}{a_1}\right) + \frac{2a_0^2 f(\tau_2 + \varphi(\tau_2)/a_0)}{a_1(a_0 + a_1)}\right\} + \frac{2(a_1 - a_0)}{a_1} f(t_0)$$

Here f^{-1} is a function reciprocal to f in the time interval $[t_s, t_0]$, where t_s is the instant when plastic strains first appear on the end of the bar. If the function $\varphi(\tau_2)$ is known on the section BD, then by substituting it into (5.1) we obtain parametric equations for the function $\varphi(t)$ on the section DE. We then continue the solution beyond the segment DE in the same manner, etc.

If the curve $x = \varphi(t)$ does not have a line with angular coefficient a_0 as its asymptote, then the solution can be continued to infinity by such segments as DE. Otherwise, the solution can be continued a finite number of steps. The downward continuation of the solution can be constructed analogously.

6. Material with delayed yielding. The dependences

$$\sigma = E\varepsilon, \quad |\varepsilon| \leqslant \varepsilon_s; \qquad \sigma = E_1\varepsilon + (E - E_1)\varepsilon_s\left(t - \frac{x}{a_0}\right), \quad |\varepsilon| > \varepsilon_s$$

for an active loading and

$$\sigma = \sigma_0 + E(\varepsilon - \varepsilon_0) = E\varepsilon + (E - E_1)(E_{s0} - \varepsilon_0)$$

for unloading from the strain ε_0 are taken in a scheme with linear hardening [10] for a material with delayed yielding as applied to the problem of elastic-plastic wave propagation in a semi-infinite bar $x \geqslant 0$. Here $x = \varphi(t)$ is the equation of the unloading wave and $\varepsilon_{s0} = \varepsilon_s(\varphi^{-1}(x) - x/a_0)$. The beginning of the unloading is determined by the requirement [11]

$$\partial\sigma / \partial t \leqslant E\varepsilon_s'(t - x/a_0)$$

If a stress $\sigma(0, t) = f(t)$ is given at the end of the bar $x < 0$, then the solution in the domain of active elastic-plastic strain is [10]

$$u_x = \frac{1}{E_1} F\left(t - \frac{x}{a_1}\right) + \varepsilon_s\left(t - \frac{x}{a_0}\right), \quad F(t) = f(t) - E\varepsilon_s(t) \qquad (6.1)$$

$$u_t = -\frac{a_1}{E_1} F\left(t - \frac{x}{a_1}\right) - a_0\varepsilon_s\left(t - \frac{x}{a_0}\right)$$

The motion in the unloading domain $x \leqslant \varphi(t)$ is determined by the equation

$$u_{tt} = a_0^2 u_{xx} + (a_0^2 - a_1^2)\frac{d}{dx}(\varepsilon_{s0} - \varepsilon_0)$$

$$a_0 = \sqrt{E/\rho}, \quad a_1 = \sqrt{E_1/\rho}$$

Repeating the reasoning of Sect. 1, we obtain the system of equations

$$(a_0 + a_1) F\left(\tau_1 - \frac{\varphi(\tau_1)}{a_1}\right) - (a_0 - a_1) F\left(\tau_2 - \frac{\varphi(\tau_2)}{a_1}\right) = 2a_1 F(t)$$

$$\tau_1 - t = \varphi(\tau_1)/a_0, \quad t - \tau_2 = \varphi(\tau_2)/a_0$$

which goes over for $\varepsilon_s = \text{const}$ into a system of equations for the unloading wave in a material without a delayed yielding when a stress is given at the end of the bar.

The procedure for determining the derivatives $\varphi'(t_0), \varphi''(t_0), \ldots, \varphi^{(n)}(t_0)$ is perfectly analogous to that described in Sects. 2 and 3. In particular, the deduction that the unloading wave is a straight line for a function $F(t)$ in the form of an n-th order parabola, is valid.

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